

The Osgood condition for stochastic partial differential equations.

Mohammud Foondun
University of Strathclyde

Eulalia Nualart
Pompeu Fabra University

Abstract

We study the following equation

$$\frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + b(u(t, x)) + \sigma \dot{W}(t, x), \quad t > 0,$$

where σ is a positive constant and \dot{W} is a space-time white noise. The initial condition $u(0, x) = u_0(x)$ is assumed to be a nonnegative and continuous function. We first study the problem on $[0, 1]$ with homogeneous Dirichlet boundary conditions. Under some suitable conditions, together with a theorem of Bonder and Groisman in [4], our first result shows that the solution blows up in finite time if and only if

$$\int_0^\infty \frac{1}{b(s)} ds < \infty,$$

which is the well-known Osgood condition. We also consider the same equation on the whole line and show that the above condition is sufficient for the nonexistence of global solutions. Various other extensions are provided; we look at equations with fractional Laplacian and spatial colored noise in \mathbf{R}^d .

Keywords: Fractional stochastic heat equation, space-time white noise, spatial colored noise.

1 Introduction and main results

Consider the following non-linear heat equation,

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + u(t, x)^{1+\eta}, & x \in \mathbf{R}^d, t > 0, \\ u(0, x) = u_0(x), \end{cases}$$

where $u_0(x)$ is a nonnegative, continuous, and bounded function. It is well known that when $0 < \eta \leq 2/d$, there is no nontrivial global solution no matter how small the nontrivial

initial condition u_0 is, while for $\eta > 2/d$, one can construct nontrivial global solutions when u_0 is small enough; see [6, 7, 10] for more precise statements and proofs. The exponent $\eta_c = 2/d$ is often called the Fujita exponent after the author of the very influential paper [6]. When the equation is considered on the interval $[0, 1]$ with homogeneous Dirichlet boundary conditions (ie. $u(t, 0) = u(t, 1) = 0$), a different picture emerges. In this case, for any $\eta > 0$, one can always construct nontrivial global solutions by taking u_0 small enough. And when u_0 is large enough, there is no global solution for any $\eta > 0$; see Theorem 17.3 of [14] for a precise statement.

One can ask whether the above phenomena still occur when one perturbs the equation with a noise term. For example, consider the following stochastic heat equation

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + u(t, x)^{1+\eta} + \dot{W}(t, x) \\ u(0, x) = u_0(x), \end{cases} \quad (1.1)$$

where \dot{W} is a space-time white noise and the initial condition $u_0(x)$ is as above. More precisely, one can ask the following two questions.

- Does there exist a Fujita exponent? Or equivalently, for which values of η one can find a nonnegative initial function so that there exist global solutions?
- For the same equation on $[0, 1]$ with homogeneous Dirichlet boundary conditions, can one take u_0 small enough so that there exist global solutions no matter what $\eta > 0$ is?

For stochastic differential equations, the answer to the analogous question is given by Feller's test for explosions; see [9, Chapter 5]. It is quite surprising that much less is known for stochastic partial differential equations. To the best of our knowledge, there are only two papers which look at these types of questions; [4] and [3]. In the first paper the authors consider the equation on $[0, 1]$ and give a negative answer to the second question above. In fact they consider the following more general equation

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + b(u(t, x)) + \sigma \dot{W}(t, x), & x \in [0, 1], t > 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1.2)$$

with homogeneous Dirichlet boundary conditions. Here $\sigma > 0$, $b : \mathbf{R} \rightarrow \mathbf{R}$ is a locally Lipschitz function, and the initial condition $u_0(x)$ is taken to be nonnegative and continuous; we will assume this throughout the whole paper. The stochastic forcing term \dot{W} is a space-time white noise. The main result of [4] says that the solution to (1.2) blows up in finite time whenever b is nonnegative, convex, and satisfies the following well-known *Osgood condition*:

$$\int_0^\infty \frac{1}{b(s)} ds < \infty. \quad (1.3)$$

In [3], the authors investigate whether the Osgood condition is optimal. In particular, their Theorem 1.4 shows that if $|b(x)| = O(|x| \log |x|)$ as $|x| \rightarrow \infty$, then there exists a global solution to equation (1.2).

As far as we know, the first question above has not been addressed till now. We briefly summarise the main findings of this current paper. For equation (1.2), we will show that Osgood condition (1.3) is also necessary. Together with the result in [4], this result shows the optimality of the Osgood condition. We will then consider equation (1.1) and answer the first question. We will show that $\eta_c = \infty$ meaning that there is no global solution no matter how small the initial condition is. This shows that the Fujita phenomenon does not occur in this stochastic setting. In fact, we will show that the Osgood condition (1.3) is sufficient for the nonexistence of global solutions for equation (1.1).

Before giving the main results of the paper, we provide some precision on various assumptions and technicalities. We will need the following condition.

Assumption 1.1. *The function $b : \mathbf{R} \rightarrow \mathbf{R}^+$ is nonnegative and nondecreasing on $(0, \infty)$.*

As in [3], we look at random field solutions.

Definition 1.2. *A local random field solution to (1.2) is a jointly measurable and adapted space-time process $u = \{u(t, x)\}_{(t,x) \in \mathbf{R}^+ \times [0,1]}$ satisfying the following integral equation*

$$\begin{aligned} u(t, x) = & \int_0^1 p(t, x, y) u_0(y) dy + \int_0^t \int_0^1 p(t-s, x, y) b(u(s, y)) dy ds \\ & + \sigma \int_0^t \int_0^1 p(t-s, x, y) W(dy ds), \end{aligned}$$

for all $t \in (0, \tau)$, where τ is some stopping time. If we can take $\tau = \infty$, then the local solution is also a global one. The function $p(t, x, y)$ is the heat kernel associated with the operator Δ with Dirichlet boundary conditions.

As u_0 is continuous and b is locally Lipschitz, the existence and uniqueness of a local solution to equation (1.2) is not an issue. Indeed, for each $N \geq 1$, one can define the truncation function

$$b_N(x) := \mathbf{1}_{\{|x| \leq N\}} b(x) + \mathbf{1}_{\{|x| > N\}} b(N) + \mathbf{1}_{\{|x| < -N\}} b(-N) \quad (1.4)$$

and obtain a unique global solution $\{u_N(t, x)\}_{(t,x) \in \mathbf{R}^+ \times [0,1]}$ to equation (1.2) where b is replaced by b_N . Moreover, $u_N(t, x)$ is almost surely continuous in (t, x) . We consider the stopping time

$$\tau_N := \inf \left\{ t > 0 : \sup_{x \in [0,1]} |u_N(t, x)| > N \right\}.$$

Then by the local property of the stochastic integral, one can easily show (see [3, Section 4]) that for each $N \geq \|u_0\|_\infty$, we have a unique local random field solution $u(t, x) = u_N(t, x)$ for all $x \in [0, 1]$ and $t \in [0, \tau_N)$. In particular, $u(t, x)$ is almost surely continuous in (t, x) . Moreover, $\tau_N \leq \tau_{N+1}$. Denote $\tau_\infty = \lim_{N \rightarrow \infty} \tau_N$. If this limit is finite, then we say that the solution blows up in finite time. Alternatively, since the noise is additive, one could also use a local inversion theorem to obtain local existence of solutions. We are now ready to state the first result of this paper.

Theorem 1.3. *Suppose that Assumption 1.1 holds. If almost surely the solution to (1.2) blows up in finite time then b satisfies the Osgood condition (1.3).*

Together with the result of Bonder and Groisman, this can be seen as an extension of a similar result for stochastic differential equations with additive noise. Indeed in later case, Feller's test for explosions says that the Osgood condition is necessary and sufficient for blow up of the solution when the noise is a Brownian motion. We will later describe a new method for proving this without appealing to Feller's test that works for a larger class of processes including the bifractional Brownian motion. This is due to [13] which was also the inspiration for the proof of the above theorem.

We also consider equation (1.2) in the whole line, that is,

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \Delta u(t, x) + b(u(t, x)) + \sigma \dot{W}(t, x) & x \in \mathbf{R}, t > 0, \\ u(0, x) = u_0(x). \end{cases} \quad (1.5)$$

As before, we look at the random field solution $u = \{u(t, x)\}_{(t,x) \in \mathbf{R}^+ \times \mathbf{R}}$ which in this case, satisfies the following integral equation

$$\begin{aligned} u(t, x) = \int_{\mathbf{R}} G(t, x, y) u_0(y) dy + \int_0^t \int_{\mathbf{R}} G(t-s, x, y) b(u(s, y)) dy ds \\ + \sigma \int_0^t \int_{\mathbf{R}} G(t-s, x, y) W(dy ds), \end{aligned} \quad (1.6)$$

where now $G(t, x, y)$ is the heat kernel associated with the Laplacian defined on the whole line. Here, the existence of a local solution is not straightforward. We will be more precise about this later. Next, we describe our second main result which is a non-existence result.

Theorem 1.4. *Suppose that Assumption 1.1 holds. Then, if b satisfies the Osgood condition (1.3), then almost surely, there is no global solution to equation (1.5).*

Here we use a completely different approach to that of [4]. We use the almost sure growth properties of the stochastic term together with an observation borrowed from [13] to

arrive at our result. This observation is contained in the statement and proof of Proposition 2.2 below. A key step in our strategy is to use the fact that for each x , the stochastic term in (1.6) is a bifractional Brownian motion. We will use various continuity estimates as well as the law of iterated logarithm for bifractional Brownian motion to arrive at the growth properties we need.

Our method is flexible enough so that some of the results above can be extended to a wider class of equations. We describe these results next. Consider the following equation

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \mathcal{L}u(t, x) + b(u(t, x)) + \dot{F}(t, x), & x \in B_1(0), t > 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1.7)$$

where $B_r(z)$ denotes the (open) ball of center z and radius r in \mathbf{R}^d . Here \mathcal{L} is the generator of an α -stable process killed upon exiting the ball $B_1(0)$ and \dot{F} is a Gaussian noise which is white in time and has a spatial correlation given by the Riesz kernel. That is,

$$\mathbb{E}(\dot{F}(t, x)\dot{F}(s, y)) = \delta_0(t - s)f(x - y),$$

where $f(x) = |x|^{-\beta}$, $0 < \beta < d$. The homogeneous Dirichlet boundary conditions is given by

$$u(t, x) = 0 \quad x \in \mathbf{R}^d \setminus B_1(0), \quad t > 0.$$

As before, the solution to equation (1.7) is a jointly measurable adapted random field $u = \{u(t, x)\}_{t \geq 0, x \in B_1(0)}$ satisfying the integral equation

$$\begin{aligned} u(t, x) = & \int_{B_1(0)} p_\alpha(t, x, y)u_0(y) dy + \int_0^t \int_{B_1(0)} p_\alpha(t - s, x, y)b(u(s, y)) dy ds \\ & + \sigma \int_0^t \int_{B_1(0)} p_\alpha(t - s, x, y)\dot{F}(dy ds), \end{aligned} \quad (1.8)$$

where $p_\alpha(t, x, y)$ is the Dirichlet fractional heat kernel.

As for equation (1.2), one can easily show that if $\beta < \alpha$, then for each $N \geq \|u_0\|_\infty$, there exists a unique local random field solution $u(t, x)$ to equation (1.7) defined for all $x \in B_1(0)$ and $t \in [0, \tau_N)$, where

$$\tau_N := \inf \left\{ t > 0 : \sup_{x \in B_1(0)} |u_N(t, x)| > N \right\},$$

and $u_N(t, x)$ is the solution to equation (1.7) where b is replaced by b_N defined in (1.4). The condition $\beta < \alpha$ ensures that the stochastic integral in (1.8) is well-defined and almost surely continuous; see Remark 1.7 below.

The next result is the extension to equation (1.7) of Bonder and Groisman theorem and of Theorem 1.3.

Theorem 1.5. *Suppose that Assumption 1.1 holds. Then if b satisfies the Osgood condition (1.3) and under the additional assumption that b is convex, the solution to (1.7) blows up in finite time almost surely. On the other hand, if the solution blows up in finite time, then b satisfies the Osgood condition (1.3).*

Remark 1.6. *The proof of the first part of Theorem 1.5 is an adaptation of the proof in [4]. But our method can do better, it can be used to prove that $\inf_{x \in B(0, 1-\epsilon)} u(t, x)$ blows up in finite time for any $\epsilon > 0$. We leave the proof for future work.*

Remark 1.7. *Theorem 1.5 holds for a general spatial correlation f , where $f : \mathbf{R}^d \rightarrow \mathbf{R}$ is a nonnegative and nonnegative definite (generalized) function, continuous on $\mathbf{R}^d \setminus \{0\}$, integrable in a neighborhood of 0, and whose Fourier transform $\mathcal{F}f = \mu$ is a tempered measure satisfying*

$$\int_{\mathbf{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^\alpha)^\rho} < \infty, \quad (1.9)$$

for some $\rho \in (0, 1)$, where $(\mathcal{F}f)(\xi) = \int_{\mathbf{R}^d} f(y)e^{i\langle y, \xi \rangle} dy$. Condition (1.9) with $\rho = 1$ implies the existence and uniqueness of solutions; see Dalang [1]. The slightly more stringent condition (1.9) ensures that the solution is almost surely continuous as well; see Sanz-Solé and Sarrà [15]. In particular, when f is the Riesz kernel, then $\mu(d\xi) = c|\xi|^{-(d-\beta)}d\xi$ and condition (1.9) holds for any $\rho > \beta/\alpha$ whenever $\beta < \alpha$.

Consider now equation (1.7) in the whole space, that is,

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \mathcal{L}u(t, x) + b(u(t, x)) + \dot{F}(t, x), & x \in \mathbf{R}^d, t > 0, \\ u(0, x) = u_0(x), \end{cases} \quad (1.10)$$

where all the parameters are the same as above except that now \mathcal{L} is associated with an α -stable process defined on the whole space.

As before, we are looking at random field solutions satisfying the following integral equation

$$\begin{aligned} u(t, x) = & \int_{\mathbf{R}^d} G_\alpha(t, x, y)u_0(y) dy + \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s, x, y)b(u(s, y)) dy ds \\ & + \sigma \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s, x, y)F(dy ds), \end{aligned} \quad (1.11)$$

where now $G_\alpha(t, x, y)$ is the heat kernel for the α -stable process. Our final theorem is as follows.

Theorem 1.8. *Suppose that Assumption 1.1 holds. Then, if b satisfies the Osgood condition (1.3), then almost surely, there is no global solution to equation (1.10).*

We end this introduction with some remarks concerning local existence of solutions when the equations are defined on the whole space. D. Khoshnevisan pointed to us that since for any fixed $t > 0$, the last term of (1.6) grows like $\sqrt{\log x}$ as x goes to infinity, the solution to (1.6) might blow up instantaneously. That is, any solution of (1.6) can blow up for any $t > 0$ so that there is no local solution. In the deterministic setting, similar phenomenon arises; see for instance [16] where the exponential reaction-diffusion is studied. Proving such non-existence results is beyond the scope of this paper where the main concern is non-existence of global solution. The above result for instance makes no claim about the existence of a local solution.

The rest of the paper is organized as follows. In Section 2 we give some preliminary results needed for the proofs of our results. Section 3 is devoted to the proofs of Theorems 1.3 and 1.4. Finally, Theorems 1.5 and 1.8 are proved in Section 4.

2 Preliminary information and estimates

In this section, we give some background information needed for the proof of our results. We start off with a deterministic result about integral equations. This is taken from [13] where it is used to show blow up for stochastic differential equations. We include a proof since it contains the main ideas of our method.

2.1 The Osgood condition for integral equations

We start off with the following remark. Suppose that b satisfies Assumption 1.1 and consider the following integral equation for $a \geq 0$

$$y(t) = a + \int_0^t b(y(s)) \, ds, \quad t \geq 0.$$

The blow up time of y is defined as

$$T := \sup\{t > 0 : |y(t)| < \infty\}.$$

Then one can show that this blow up time is equal to the following

$$\int_a^\infty \frac{1}{b(s)} \, ds.$$

We next consider the following assumption. In the upcoming sections, we will show that a large class of stochastic processes verify a similar condition.

Assumption 2.1. $g : [0, \infty) \rightarrow \mathbf{R}$ is a continuous function such that

$$\limsup_{t \rightarrow \infty} \inf_{0 \leq h \leq 1} g(t+h) = \infty.$$

Proposition 2.2. *Let $a \geq 0$ and suppose that Assumptions 1.1 and 2.1 hold. Then the solution to the integral equation*

$$X_t = a + \int_0^t b(X_s) ds + g(t) \quad (2.1)$$

blows up in finite time if and only if the function b satisfies the Osgood condition (1.3).

Proof. Suppose that the solution blows up at finite time T . Since g is continuous, we can set

$$M := \sup_{0 \leq s \leq T} |g(s)|.$$

Let $0 \leq t \leq T$. Upon noting that b is nonnegative, (2.1) gives

$$X_t \leq a + M + \int_0^t b(X_s) ds.$$

The nonnegativity of b together with the continuity of g imply that X_t can only blow up to *positive infinity*. Let $Y_t = a + M + 1 + \int_0^t b(Y_s) ds$. Then by a standard comparison result, we have $X_t \leq Y_t$ on $[0, T]$. But since X_t blows up at time T , Y_t should also blow up by time T . This means that b satisfies the Osgood condition (1.3).

We now suppose that X_t does not blow up in finite time. Let $\{t_n\}_{n=1}^\infty$ be some sequence which tends to infinity. The nonnegativity of b implies that

$$\begin{aligned} X_{t+t_n} &\geq a + \int_{t_n}^{t+t_n} b(X_s) ds + g(t+t_n) \\ &\geq a + \int_0^t b(X_{s+t_n}) ds + g(t+t_n) \\ &\geq \frac{1}{2} \left(a + \inf_{0 \leq h \leq 1} g(h+t_n) \right) + \int_0^t b(X_{s+t_n}) ds, \end{aligned}$$

where the last inequality holds whenever $0 \leq t \leq 1$. This means that $X_{t+t_n} \geq Z_t$ where

$$Z_t = \frac{1}{2} \left(a + \inf_{0 \leq h \leq 1} g(h+t_n) \right) + \int_0^t b(Z_s) ds.$$

Since we are assuming that X_t does not blow up in finite time, the blow up time of Z_t has to be greater than 1, which implies that

$$\int_{\frac{1}{2}(a + \inf_{0 \leq h \leq 1} g(h+t_n))}^\infty \frac{1}{b(s)} ds > 1.$$

But from Assumption 2.1, we can find a sequence $t_n \rightarrow \infty$ such that $\frac{1}{2}(a + \inf_{0 \leq h \leq 1} g(h + t_n)) \rightarrow \infty$. Hence

$$\int_{\cdot}^{\infty} \frac{1}{b(s)} ds = \infty,$$

and the proof is complete. \square

As mentioned in the introduction, the above result provides an alternative way to prove blow-up for stochastic differential equations of the following type,

$$dX_t = b(X_t) dt + dB_t, \quad X_0 = a,$$

where B_t is a Brownian motion. This can be written as the following integral equation,

$$X_t = a + \int_0^t b(X_s) ds + B_t.$$

We can now show that almost surely B_t satisfies Assumption 2.1 above. Hence the Osgood condition is a necessary and sufficient condition for blow-up of the solution to the above equation. As showed in [13], one can replace the Brownian motion by a more general class of processes including the bifractional Brownian motion for which Feller's test for explosions is not applicable.

2.2 The bifractional Brownian motion and related results

The bifractional Brownian motion introduced in [8] is a generalization of the fractional Brownian motion. It is defined as a centered Gaussian process $B^{H,K} = (B_t^{H,K}, t \geq 0)$ with covariance

$$R^{H,K}(t, s) = 2^{-K} ((t^{2H} + s^{2H})^K - |t - s|^{2HK}),$$

where $H \in (0, 1)$ and $K \in (0, 1]$. Note that if $K = 1$, then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter H . The bifractional Brownian motion is Hölder continuous for any exponent less than HK . Moreover, it satisfies the following law of iterated logarithm; see for instance Lemma 4.1 of [13] for an idea of the proof and further references. Set

$$\psi_{H,K}(t) := t^{HK} \sqrt{2 \log \log t}, \quad t > e.$$

Lemma 2.3. *Almost surely,*

$$\limsup_{t \rightarrow \infty} \frac{B_t^{H,K}}{\psi_{H,K}(t)} = 1.$$

Consider now the process

$$g(t, x) := \int_0^t \int_{\mathbf{R}} p(t - s, x, y) W(dy ds),$$

where we recall that $p(t, x, y)$ denotes the Gaussian heat kernel. Clearly the above is the solution to the stochastic heat equation (1.5) with zero drift, zero initial condition, and $\sigma = 1$.

It is shown in [12] that for a fixed $x \in \mathbf{R}$, the process $(g(t, x), t \geq 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$ multiplied by a constant. In fact, the covariance of $g(t, x)$ is given by

$$\mathbb{E}(g(t, x)g(s, x)) = \frac{1}{\sqrt{2\pi}}(\sqrt{t+s} - \sqrt{|t-s|}).$$

In particular, the process $(g(t, x), t \geq 0)$ is Hölder continuous for any exponent less than $HK = 1/4$.

The following estimates on the increments of $g(t, x)$ are well known. For instance see Theorem 6.7 of [5] and the proof of Corollary 3.4 in [17].

Lemma 2.4. *For all $p \geq 2$ there exist constants $c_p, \tilde{c}_p > 0$ such that for all $x, y \in \mathbf{R}$ and $s, t \geq 0$,*

$$\sup_{t \geq 0} \mathbb{E}[|g(t, x) - g(t, y)|^p] \leq c_p |x - y|^{p/2}$$

and

$$\sup_{x \in \mathbf{R}} \mathbb{E}[|g(t, x) - g(s, x)|^p] \leq \tilde{c}_p |s - t|^{p/4}.$$

As a consequence of Lemma 2.4 and the improvement of the classical Garsia's lemma obtained in Proposition A.1. of [2], we have the following estimate.

Proposition 2.5. *For all $p \geq 2$, there exists a constant $A_p > 0$ such that for any integer $n \geq 1$,*

$$\mathbb{E} \left[\sup_{s, t \in [n, n+2], x, y \in [0, 1]} |g(t, x) - g(s, y)|^p \right] \leq A_p 2^{p/4}.$$

Proof. The proof follows along the same lines as the proof of Lemma 4.5 in [2]. Indeed, by Lemma 2.4 and Proposition A.1. in [2], we get that for all $p \geq 2$, there exists a constant $A_p > 0$ such that for any $\epsilon > 0$,

$$\mathbb{E} \left[\sup_{(|t-s|^{1/2} + |x-y|)^{1/2} \leq \epsilon} |g(t, x) - g(s, y)|^p \right] \leq A_p \epsilon^p.$$

Then, using this inequality with $\epsilon = \sqrt{2}2^{1/4}$ implies the desired result. \square

We can now use Proposition 2.5 to get the following almost sure result. This is an extension to the multiparameter case of Lemma 4.2 in [13].

Proposition 2.6. *Almost surely,*

$$\sup_{s,t \in [n, n+2], x,y \in [0,1]} \frac{|g(t, x) - g(s, y)|}{\psi_{\frac{1}{2}, \frac{1}{2}}(n)} \longrightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proof. Proposition 2.5 implies that

$$\mathbb{E} \left[\sum_{n=1}^{\infty} \sup_{s,t \in [n, n+2], x,y \in [0,1]} \frac{|g(t, x) - g(s, y)|^p}{\psi_{\frac{1}{2}, \frac{1}{2}}(n)^p} \right] \leq \sum_{n=1}^{\infty} \frac{A_p 2^{p/4}}{\psi_{\frac{1}{2}, \frac{1}{2}}(n)^p} < \infty,$$

which gives us the desired result. \square

As a consequence of Proposition 2.6, we get the following estimate.

Proposition 2.7. *Almost surely, there exists a sequence t_n such that*

$$\inf_{h \in [0,1], x \in [0,1]} g(t_n + h, x) \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Proof. Fix $x_0 \in [0, 1]$. Choose ω such that both Proposition 2.6 and Lemma 2.3 hold. We now write

$$\begin{aligned} \inf_{h \in [0,1], x \in [0,1]} g(t + h, x) &= g(t, x_0) + \inf_{h \in [0,1], x \in [0,1]} (g(t + h, x) - g(t, x_0)) \\ &\geq g(t, x_0) + \inf_{h \in [0,1], x \in [0,1]} (-|g(t + h, x) - g(t, x_0)|) \\ &\geq \frac{g(t, x_0)}{\psi_{\frac{1}{2}, \frac{1}{2}}(t)} \psi_{\frac{1}{2}, \frac{1}{2}}(t) - \sup_{h \in [0,1], x \in [0,1]} \frac{|g(t + h, x) - g(t, x_0)|}{\psi_{\frac{1}{2}, \frac{1}{2}}([t])} \psi_{\frac{1}{2}, \frac{1}{2}}([t]). \end{aligned}$$

We use Proposition 2.6 and Lemma 2.3 to choose an appropriate sequence t_n and finish the proof. \square

3 Proofs of Theorems 1.3 and 1.4

As mention earlier, the proof of Theorem 1.3 follows that of Proposition 2.2 but it heavily relies on the fact that the stochastic term in the random field formulation is continuous and that the equation itself is defined on an interval.

Proof of Theorem 1.3. Set

$$T := \sup\{t > 0 : \sup_{x \in [0,1]} |u(t, x)| < \infty\} < \infty.$$

Since the solution blows up almost surely, we can find a set Ω satisfying $P(\Omega) = 1$ such that for any $\omega \in \Omega$, we have $T(\omega) < \infty$. We now fix such an ω but for the sake of notational

convenience, we won't indicate the dependence on ω in what follows. We recall that we are looking at the mild formulation

$$\begin{aligned} u(t, x) = & \int_0^1 p(t, x, y) u_0(y) dy + \int_0^t \int_0^1 p(t-s, x, y) b(u(s, y)) dy ds \\ & + \sigma \int_0^t \int_0^1 p(t-s, x, y) W(dy ds). \end{aligned}$$

The third term in the above display is almost surely continuous. Therefore the following quantity below is finite almost surely

$$M := \sup_{x \in [0, 1] t \in (0, T]} \left| \int_0^t \int_0^1 p(t-s, x, y) W(dy ds) \right|.$$

Moreover by the nonnegativity of b and the initial condition, we have

$$u(t, x) \geq \int_0^t \int_0^1 p(t-s, x, y) W(dy ds).$$

This means that

$$\inf_{t \in [0, T], x \in [0, 1]} u(t, x) \geq -\tilde{M},$$

where \tilde{M} is finite almost surely. Since u_0 is bounded, we have

$$\left| \int_0^1 p(t, x, y) u_0(y) dy \right| \leq c,$$

for some positive constant c . Denote $\mathcal{A} := \{s \in (0, t), y \in (0, 1); -\tilde{M} \leq u(s, y) \leq 0\}$ and $\mathcal{B} := \{s \in (0, t), y \in (0, 1); u(s, y) > 0\}$ and write

$$\begin{aligned} \int_0^t \int_0^1 p(t-s, x, y) b(u(s, y)) dy ds &= \iint_{\mathcal{A}} p(t-s, x, y) b(u(s, y)) dy ds \\ &+ \iint_{\mathcal{B}} p(t-s, x, y) b(u(s, y)) dy ds \\ &:= I_1 + I_2. \end{aligned}$$

Since we are assuming that b is nonnegative and nondecreasing on $(0, \infty)$, this immediately gives us

$$I_2 \leq \int_0^t b(Y_s) ds,$$

where $Y_t := \sup_{x \in [0, 1]} u(t, x)$. Since b is assumed to be continuous, we have $I_1 \leq K$, where K is an almost sure finite quantity. Putting all these estimates together, we obtain

$$Y_t \leq a + \sigma M + K + \int_0^t b(Y_s) ds.$$

We can now proceed as in the proof of Proposition 2.2 to conclude the proof. \square

Remark 3.1. We remark that the definition of the blow up time T used above is not the same as τ_∞ . But one can show that $T < \infty$ if and only if $\tau_\infty < \infty$. To see this one can for instance look at $u(\tau_N, x) = N$ and by letting $N \rightarrow \infty$ and using the definition of T , we arrive at the required conclusion. Indeed if $T < \infty$, then that would mean that $\tau_\infty < \infty$. While if $T = \infty$, then one can see that τ_∞ cannot be finite.

Proof of Theorem 1.4. Let $\{t_n\}$ be a sequence of positive numbers which we are going to choose later. From the mild formulation of the solution and the nonnegativity of the function b , we obtain

$$\begin{aligned} u(t + t_n, x) &= \int_{\mathbf{R}} p(t + t_n, x, y) u_0(y) dy + \int_0^{t+t_n} \int_{\mathbf{R}} p(t + t_n - s, x, y) b(u(s, y)) dy ds \\ &\quad + \sigma \int_0^{t+t_n} \int_{\mathbf{R}} p(t + t_n - s, x, y) W(dy ds) \\ &\geq \int_{\mathbf{R}} p(t + t_n, x, y) u_0(y) dy + \int_0^t \int_{\mathbf{R}} p(t - s, x, y) b(u(s + t_n, y)) dy ds \\ &\quad + \sigma \int_0^{t+t_n} \int_{\mathbf{R}} p(t + t_n - s, x, y) W(dy ds). \end{aligned}$$

We will take $0 \leq t \leq 1$. Recall that

$$g(t + t_n, x) := \int_0^{t+t_n} \int_{\mathbf{R}} p(t + t_n - s, x, y) W(dy ds).$$

Hence by Proposition 2.7, we can find a sequence $\{t_n\}$ so that the above quantity is positive for $0 \leq t \leq 1$. Therefore $u(t + t_n, x)$ is also positive for any $x \in (0, 1)$ and any $0 \leq t \leq 1$. We now use the fact that b is nondecreasing on $(0, \infty)$ to bound the second term as follows. For fixed $x \in (0, 1)$,

$$\begin{aligned} \int_0^t \int_{\mathbf{R}} p(t - s, x, y) b(u(s + t_n, y)) dy ds \\ &\geq \int_0^t b \left(\inf_{y \in (0, 1)} u(s + t_n, y) \right) \int_{(0, 1)} p(t - s, x, y) dy ds \\ &\geq \int_0^t b \left(\inf_{y \in (0, 1)} u(s + t_n, y) \right) ds, \end{aligned}$$

where we have used that fact that $p_t(x, y) \geq \frac{c}{t^{1/2}}$ whenever $|x - y| \leq t^{1/2}$. We now set $Y_t := \inf_{y \in (0, 1)} u(t + t_n, y)$ and combine the above estimates to obtain

$$Y_t \geq \inf_{0 \leq h \leq 1, x \in (0, 1)} \left\{ \int_{\mathbf{R}} p(h + t_n, x, y) u_0(y) dy + \sigma g(h + t_n, x) \right\} + \int_0^t b(Y_s) ds.$$

We now choose ω as in Proposition 2.7, and we can therefore find a sequence $t_n \rightarrow \infty$ such that $\inf_{0 \leq h \leq 1, x \in (0, 1)} g(h + t_n, x)$ goes to infinity. By the proof of Proposition 2.2, we have the required result. \square

4 Extension to fractional Laplacian and colored noise

The aim of this section is to prove Theorems 1.5 and 1.8. For this, we first define rigorously the Gaussian noise F and extend the results of Section 2.2 to the equation in \mathbf{R}^d .

4.1 The Gaussian noise F

Let $\mathcal{D}(\mathbf{R}_+ \times \mathbf{R}^d)$ be the space of real-valued infinitely differentiable functions with compact support. Following [1], on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, we consider a centered Gaussian family of random variables $\{F(\varphi), \varphi \in \mathcal{D}(\mathbf{R}_+ \times \mathbf{R}^d)\}$ with covariance

$$\mathbb{E}[F(\varphi)F(\psi)] = \int_{\mathbf{R}_+ \times \mathbf{R}^{2d}} \varphi(t, x) \psi(t, y) f(x - y) dx dy dt,$$

where f is as in Remark 1.7. Let \mathcal{H} be the completion of $\mathcal{D}(\mathbf{R}_+ \times \mathbf{R}^d)$ with respect to the inner product

$$\begin{aligned} \langle \varphi, \psi \rangle_{\mathcal{H}} &= \int_{\mathbf{R}_+ \times \mathbf{R}^{2d}} \varphi(t, x) \psi(t, y) f(x - y) dx dy dt \\ &= \int_{\mathbf{R}_+ \times \mathbf{R}^d} \mathcal{F}\varphi(t, \cdot)(\xi) \overline{\mathcal{F}\psi(t, \cdot)(\xi)} \mu(d\xi) dt, \end{aligned} \tag{4.1}$$

where the last equality follows by Parseval's identity. The mapping $\varphi \mapsto F(\varphi)$ defined in $\mathcal{D}(\mathbf{R}_+ \times \mathbf{R}^d)$ extends to a linear isometry between \mathcal{H} and the Gaussian space spanned by F . We will denote the isometry by

$$F(\varphi) = \int_{\mathbf{R}_+ \times \mathbf{R}^d} \varphi(t, x) F(dt dx), \quad \varphi \in \mathcal{H}.$$

Notice that if $\varphi, \psi \in \mathcal{H}$, then $\mathbb{E}[F(\varphi)F(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Moreover, \mathcal{H} contains the space of measurable functions ϕ on $\mathbf{R}_+ \times \mathbf{R}^d$ such that

$$\int_{\mathbf{R}_+ \times \mathbf{R}^{2d}} |\phi(t, x) \phi(t, y)| f(x - y) dx dy dt < \infty.$$

4.2 Estimates for the whole line

Consider the solution to the stochastic heat equation (1.10) with zero drift, zero initial condition, and $\sigma = 1$, that is,

$$g_{\alpha,\beta}(t, x) := \int_0^t \int_{\mathbf{R}^d} G_\alpha(t-s, x, y) F(dy ds),$$

where recall that $G_\alpha(t, x, y)$ is the fractional heat kernel and F has a spatial correlation given by the Riesz kernel.

Let us compute the covariance of the Gaussian process $(g_{\alpha,\beta}(t, x), t \geq 0)$ for $x \in \mathbf{R}^d$ fixed. By (4.1), as

$$\mathcal{F}G_\alpha(t, x, \cdot)(\xi) = e^{i\langle x, \xi \rangle - \frac{1}{2}t|\xi|^\alpha}, \quad \xi \in \mathbf{R}^d, \quad (4.2)$$

we get that, for $s \leq t$,

$$\begin{aligned} & \mathbb{E}(g_{\alpha,\beta}(t, x)g_{\alpha,\beta}(s, x)) \\ &= \int_0^s \int_{\mathbf{R}^d \times \mathbf{R}^d} G_\alpha(t-u, x, y)G_\alpha(s-u, x, z)|z-y|^{-\beta} dy dz du \\ &= c_{d,\beta} \int_0^s \int_{\mathbf{R}^d} |\xi|^{-(d-\beta)} e^{-\frac{1}{2}(t-u)|\xi|^\alpha} e^{-\frac{1}{2}(s-u)|\xi|^\alpha} d\xi du \\ &= c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{-(d-\beta)-\alpha} e^{-\frac{1}{2}(t+s)|\xi|^\alpha} \left(e^{s|\xi|^\alpha} - 1 \right) d\xi \\ &= c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{-(d-\beta)-\alpha} \left(e^{-\frac{1}{2}(t-s)|\xi|^\alpha} - e^{-\frac{1}{2}(t+s)|\xi|^\alpha} \right) d\xi \\ &= c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{-(d-\beta)-\alpha} \left(\int_{-\frac{1}{2}(t+s)|\xi|^\alpha}^0 e^z dz - \int_{-\frac{1}{2}(t-s)|\xi|^\alpha}^0 e^z dz \right) d\xi \\ &= c_{d,\beta,\alpha} \left((t+s)^{1-\frac{\beta}{\alpha}} - (t-s)^{1-\frac{\beta}{\alpha}} \right), \end{aligned}$$

where $c_{d,\beta,\alpha} = c_{d,\beta} \int_{\mathbf{R}^d} |\xi|^{-(d-\beta)-\alpha} (1 - e^{-\frac{1}{2}|\xi|^\alpha}) d\xi$.

Therefore, for $x \in \mathbf{R}^d$ fixed, the process $(g(t, x), t \geq 0)$ is a bifractional Brownian motion with parameters $H = \frac{\alpha-\beta}{2}$ and $K = \frac{1}{\alpha}$, multiplied by a constant. In particular, it is Hölder continuous for any exponent less than $HK = \frac{\alpha-\beta}{2\alpha}$.

The next proposition is the extension of Lemma 2.4 and Propositions 2.5, 2.6 and 2.7 to the process $g_{\alpha,\beta}$.

Proposition 4.1. (a) *For all $p \geq 2$ there exists constants $c_p, \tilde{c}_p > 0$ such that for all $x, y \in \mathbf{R}^d$ and $s, t \geq 0$,*

$$\sup_{t \geq 0} \mathbb{E} [|g_{\alpha,\beta}(t, x) - g_{\alpha,\beta}(t, y)|^p] \leq c_p |x - y|^{\frac{(\alpha-\beta)p}{2}}$$

and

$$\sup_{x \in \mathbf{R}^d} \mathbb{E} [|g_{\alpha,\beta}(t, x) - g_{\alpha,\beta}(s, x)|^p] \leq \tilde{c}_p |s - t|^{\frac{(\alpha-\beta)p}{2\alpha}}.$$

(b) For all $p \geq 2$, there exists a constant $A_p > 0$ such that for any integer $n \geq 1$,

$$\mathbb{E} \left[\sup_{s, t \in [n, n+2], x, y \in B_1(0)} |g_{\alpha,\beta}(t, x) - g_{\alpha,\beta}(s, y)|^p \right] \leq A_p 2^{\frac{(\alpha-\beta)p}{2\alpha}}.$$

(c) Almost surely,

$$\sup_{s, t \in [n, n+2], x, y \in B_1(0)} \frac{|g_{\alpha,\beta}(t, x) - g_{\alpha,\beta}(s, y)|}{\psi_{\frac{\alpha-\beta}{2}, \frac{1}{\alpha}}(n)} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

(d) Almost surely, there exists a sequence t_n such that

$$\inf_{h \in [0, 1], x \in B_1(0)} g_{\alpha,\beta}(t_n + h, x) \rightarrow \infty.$$

Proof. The proof of (a) follows from [15] and (4.2). Moreover, (a) implies that Lemma 4.5 in [2] also holds for our process $g_{\alpha,\beta}(t, x)$, which gives (b). Finally, (c) and (d) follow as in the proof of Propositions 2.6 and 2.7. \square

4.3 Proof of Theorems 1.5 and 1.8

The proof of Theorem 1.5 follows closely the proof of the main result in [4]. Since, our equation is different, we present a brief proof here.

Proof of Theorem 1.5. Recall the random field formulation of the solution to equation (1.7)

$$\begin{aligned} u(t, x) = & \int_{B_1(0)} p_\alpha(t, x, y) u_0(y) dy + \int_0^t \int_{B_1(0)} p_\alpha(t-s, x, y) b(u(s, y)) dy ds \\ & + \sigma \int_0^t \int_{B_1(0)} p_\alpha(t-s, x, y) F(dy ds), \end{aligned} \tag{4.3}$$

where $p_\alpha(t, x, y)$ is the Dirichlet fractional heat kernel. It is well known that we have the following spectral decomposition

$$p_\alpha(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y) \quad \text{for all } x, y \in B_1(0), \quad t > 0.$$

The sequence of functions $\{\phi_n\}_{n \geq 1}$ is an orthonormal basis of $L^2(B_1(0))$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ is a sequence of positive numbers such that, for every $n \geq 1$,

$$\begin{cases} -(-\Delta)^{\alpha/2} \phi_n(x) = -\lambda_n \phi_n(x) & x \in B_1(0), \\ \phi_n(x) = 0 & x \in \mathbf{R}^d \setminus B_1(0). \end{cases}$$

We now set

$$Y_t = \int_{B_1(0)} u(t, x) \phi_1(x) dx.$$

Then after using the stochastic Fubini theorem and the decomposition above, (4.3) gives us

$$Y_t = e^{-\lambda_1 t} \left(Y_0 + \int_0^t e^{\lambda_1 s} \int_{B_1(0)} \phi_1(y) b(u(s, y)) dy ds + \int_0^t e^{\lambda_1 s} \int_{B_1(0)} \phi_1(y) F(dy ds) \right).$$

Since b is convex, by Jensen's inequality, we get

$$\int_{B_1(0)} \phi_1(y) b(u(s, y)) dy \geq b(Y_s).$$

Moreover,

$$\int_0^t \int_{B_1(0)} \phi_1(y) F(dy ds) = \sqrt{\kappa} B_t,$$

where B_t is a Brownian motion and

$$\kappa := \int_{B_1(0) \times B_1(0)} \phi_1(y) \phi_1(z) |y - z|^{-\beta} dy dz.$$

Therefore, we obtain that

$$e^{\lambda_1 t} Y_t \geq Y_0 + \int_0^t e^{\lambda_1 s} b(Y_s) ds + \sqrt{\kappa} \int_0^t e^{\lambda_1 s} dB_s.$$

In particular, Y_t verifies the stochastic differential inequality

$$dY_t \geq (-\lambda_1 Y_t + b(Y_t)) dt + \sqrt{\kappa} dB_t.$$

We now use Feller's test for explosion as in [4] to finish the proof. \square

The proof of Theorem 1.8 is similar to that of Theorem 1.4. We will indicate the differences only.

Proof of Theorem 1.8. The proof follows that of Theorem 1.4. We use the last part of Proposition 4.1 together with the following inequality

$$G_\alpha(t, x, y) \geq \frac{\text{constant}}{t^{d/\alpha}} \quad \text{whenever} \quad |x - y| \leq t^{1/\alpha} \quad (4.4)$$

to prove the required result. See for instance [11] for justifications of the above inequality. We leave it to the reader to fill in the details. \square

References

- [1] Robert C. Dalang. Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s. *Electron. J. Probab.*, 4:no. 6, 29 pp. (electronic), 1999.
- [2] Robert C. Dalang, Davar Khoshnevisan, and Eulalia Nualart. Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.*, 3:231–271, 2007.
- [3] Robert C. Dalang, Davar Khoshnevisan, and Tusheng Zhang. Global solutions to stochastic reaction-diffusion equations with super-linear drift and multiplicative noise. *Ann. Probab.*, to appear.
- [4] Julian Fernández Bonder and Pablo Groisman. Time-space white noise eliminates global solutions in reaction-diffusion equations. *Phys. D*, 238(2):209–215, 2009.
- [5] Mohammud Foondun, Davar Khoshnevisan, and Eulalia Nualart. A local time correspondence for stochastic partial differential equations. *Trans. Amer. Math. Soc.*, 363(5):2481–2515, 2011.
- [6] Hiroshi Fujita. On the blowing up of solutions of the Cauchy problem for $u_t = \delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo*, 13:109–124, 1966.
- [7] Kantaro Hayakawa. On nonexistence of global solutions of some semilinear parabolic equations. *Proc. Japan Acad.*, 49:503–525, 1973.
- [8] Christian Houdré and José Villa. An example of infinite dimensional quasi-helix. *Contemp. Maths*, 336:195–201, 2003.
- [9] Ioannis Karatzas and Steven E. Shreve. Brownian motion and stochastic calculus. volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [10] Kusuo Kobayashi, Tunekiti Sirao, and Hiroshi Tanaka. On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan*, 29:407–424, 1977.
- [11] Vassili Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc. (3)*, 80(3):725–768, 2000.
- [12] Pedro Lei and David Nualart. A decomposition of the bifractional Brownian motion and some applications. *Statist. Probab. Lett.*, 79:619–624, 2009.

- [13] Jorge A. León and José Villa. An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise. *Statist. Probab. Lett.*, 81:470–477, 2011.
- [14] Pavol Quittner and Philippe Souplet. *Superlinear parabolic problems. Blow-up, global existence and steady states*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, 2007.
- [15] Marta Sanz-Solé and Mònica Sarrà. Hölder continuity for the stochastic heat equation with spatially correlated noise. In *Progress in Probability*, volume 52, Seminar on Stochastic Analysis, Random Fields and Applications III, pages 259–268, 2002.
- [16] Juan Lui Vasquez. Domain of existence and blowup for the exponential reaction-diffusion equation. *Indiana Univ. Math. J.*, 48(2):677–709, 1999.
- [17] John B. Walsh. An introduction to stochastic partial differential equations. In *École d’été de Probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.